

Connecting Through Obstruction; Relating Gauge Gravity and String Theory

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We consider topological constraints that must be satisfied by formulations of gravitation as a gauge theory. To facilitate the analysis we review and further justify the composite bundle formalism of Tresguerres as a consistent underlying structure capable of incorporating both the local Lorentz and translational degrees of freedom. Identifying an important global structure required by the composite construction, we translate this into conditions on the underlying manifold. We find that in addition to admitting the expected orientability, causality and spin structures, the underlying manifold must also admit a string structure. We take this to imply that even before considerations of quantum consistency, topological considerations of gauge gravity provide a classical motivation for extended degrees of freedom.

Keywords: Gauge Gravity, Composite Fiber Bundle, String Theory

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I. INTRODUCTION

The history of attempts to formulate gravity as a gauge theory began shortly after the seminal work of Yang and Mills [1]. In the following year Utiyama made the first attempt to obtain Einstein’s field equations from a gauge principle by localizing the Lorentz symmetry of flat spacetime [2]. His efforts elucidated the complications of gauging external (or spacetime) symmetry groups, but fell short of providing a fully consistent derivation of general relativity from a gauge principle. Among its shortcomings were the ad hoc introduction of local Lorentz frames and the absence of a conserved energy-momentum tensor acting as a source of curvature. In 1961 Kibble, identifying spacetime translations as the generator of the energy-momentum current, extended Utiyama’s analysis to a gauge theory based on the full Poincaré group [3]. Kibble’s construction, often referred to as Poincaré gauge theory, reproduced the field equations, in particular realizing curvature sourced by energy-momentum, however it also required an additional torsion term sourced by the

spin-angular momentum of fields and did not provide a consistent interpretation of the role coframes. Perhaps surprisingly, in the fifty years since Kibble’s work there has yet to evolve a consensus on whether and how gravitation is realized from a gauge principle [4–12].

While much of the work on gravitation as a gauge theory has proceeded in terms of action functionals, less effort has aimed at finding an underlying construction in terms of a principal fiber bundle. This despite the necessity of the bundle formalism for addressing gauge theories in topologically nontrivial spacetimes. An immediate obstacle to any such construction based on Poincaré symmetry is clear. The Lorentz subgroup acts on the tangent-space indices of fields at a given point in spacetime and hence can be realized as fiberwise or “vertical” transformations which leave the base-point unchanged. However elements of the translation subgroup necessarily move between points in the base. This presents a difficulty in using the usual interpretation of gauge transformations as vertical fiber automorphisms. It was suggested by Lord that these could be accommodated by allowing “horizontal” components of the fiber action [7]. This idea was given a more complete realization in the work of Tresguerres who utilized the formalism of composite bundles [8]. We view the formulation due to Tresguerres as the most promising context in which to investigate the consequences of a gauge principle for spacetime topology. Investigating these matters, we find several restrictions on spacetime that might have been expected, i.e. that it be orientable and admit both spin and causal structures. However a further finding came as something of a surprise. The consistency of the composite bundle formulation seems to imply that the underlying spacetime also admit a string structure. This of course is the relevant restriction on the target spaces of perturbative string models.

The outline of this paper is as follows. We begin with a review of the standard fiber bundle formalism as it applies to gauge theory in order to establish notation. We then review the composite bundle formalism and in particular how it is applied to gauge theories of gravitation. Utilizing a consistency condition from composite fiber

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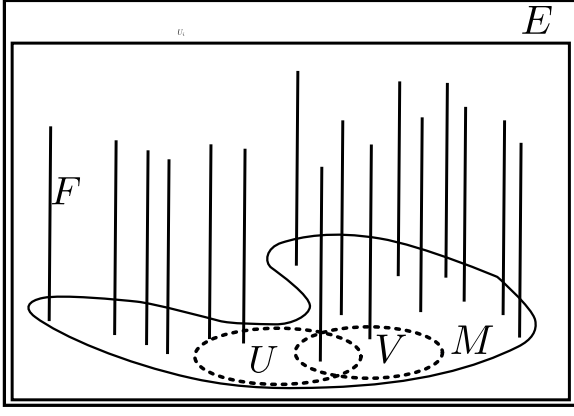


FIG. 1. A fiber bundle is shown. The base M is a differentiable manifold which we cover by open sets, where we have displayed two overlapping open neighborhoods in the cover. For each point of the neighborhoods, and so the whole manifold, we attach a fiber space F . The collection of all of the fiber spaces over all of the open neighborhoods is what we call E the total space.

bundles, we then investigate implications for the structure of the base spacetime and enumerate our findings. We conclude with comments on the significance of these findings and directions for future work.

II. FIBER BUNDLES AND GAUGE THEORIES OF INTERNAL SYMMETRIES

In what follows we follow closely the expositions in Nakahara and Frankel [13, 14]. A differentiable fiber bundle (Fig. 1) denoted by $E \xrightarrow{\pi} M$ or $E(M, F, G, \pi)$ consists of the following data,

1. Differentiable manifolds E, M and F , called the total, base and fiber space respectively.
2. A surjection $\pi : E \rightarrow M$ called the projection.
3. A Lie group G called the structure group such that G has left action on the fiber.
4. An open cover $\{U_i\}$ of the base with diffeomorphism $\phi_i : U_i \times F \rightarrow \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p \in M$.
5. On every non empty overlapping set of neighborhoods $U_i \cap U_j$ we require a G valued transition function $t_{ij} = \phi_i \circ \phi_j^{-1}$ such that $\phi_i = t_{ij} \phi_j$.

Of particular relevance for gauge theory are principal fiber bundles wherein the fiber space is taken to be the structure group G itself denoted $P \xrightarrow{\pi} M$ or $P(M, G)$. In addition to the left group action of G on the fibers we can also define a right action R_g such that if $u \in P$ we have $R_g u = ug$. As an example take the principal bundle as $P(\mathbb{R}^{1,3}, U(1))$. The group operation is multiplication and an arbitrary element of $U(1)$ can be expressed as $e^{i\theta}$.

$\mathbb{R}^{1,3}$ is a trivial base space so we need only one coordinate chart $(U, \varphi = x^\mu)$. We can write the local trivialization of a point $u \in P$ as $\phi^{-1}(u) = (p, e)$ where $p \in \mathbb{R}^{1,3}$ and $e \in U(1)$ is the identity element of the group. This is known as the canonical local trivialization. Given two trivializations ϕ, ϕ' there exists an element of G such that $\phi' = \phi g = \phi(ug)$. Therefore we have,

$$\phi'(p, e^{i\theta}) = \phi(p, e) e^{i\theta} = \phi(p, e e^{i\theta}) = \phi(p, e^{i\theta}). \quad (1)$$

By repeated application of the right action we move through the fiber G . This leads us to define the whole fiber as $\pi^{-1}(p) = \{ug | g \in G\}$. This property will hold in any principal bundle. Naturally we will always choose to represent our sections as right multiplication of the canonical local trivialization.

An important concept is that of a section of a fiber bundle. A section $\sigma : M \rightarrow E$ is a smooth mapping such that $\pi \circ \sigma(p) = id_M$. If a section is defined only on a chart U_i then we say σ_i is a local section. Sections are assignments of points in the base space with points in the total space P and induce mappings of vectors in the base space to vectors in the total space. An example of a section of a principal bundle $P(M, G)$ for $p \in M$, $e \in G$ is given by $\sigma(p) = \phi(p, e) = u \in P$.

A fiber bundle is in general an extension of the Cartesian product between two spaces X and Y , $X \times Y \equiv \{(x, y) | x \in X, y \in Y\}$. A non-trivial fiber bundle then cannot be realized as a standard Cartesian product globally. However locally we do have a trivialization of a bundle $P(M, G)$ as $M \times G$ so it is typical to work with local sections defined on subsets of the space M . Indeed a bundle will not always admit a global section, a mapping defined for the whole space rather than some subset of the cover. In fact the triviality of a principal bundle is reflected in whether or not it admits a global section. For vector bundles on the other hand which always admit a global zero section, their triviality is indicated by the presence of an everywhere non-vanishing global section. Otherwise the triviality of a vector bundle can be assessed by investigating global sections of its associated principal bundle. If the associated principal bundle has non-vanishing global sections then so does the vector bundle.

Another important consideration is the transport of a vector through the total bundle space. Of concern is if the vector in the bundle is parallel with the base or with the fiber. This can be accomplished by defining a splitting of the bundle tangent space $T_u P$ into vertical and horizontal subspaces $V_u P$ and $H_u P$ respectively (Fig. 2). The requirements of the split are,

1. $T_u P = H_u P \oplus V_u P$.
2. A smooth vector field X in P can be written as $X = X^H + X^V$ for $X^H \in H_u P$ and $X^V \in V_u P$.
3. $R_g^* H_u P = H_{ug} P$ where R_g^* is differential map induced by the right action of g .

The first two conditions specify the form of the split and the third condition comes from the idea that these fibers

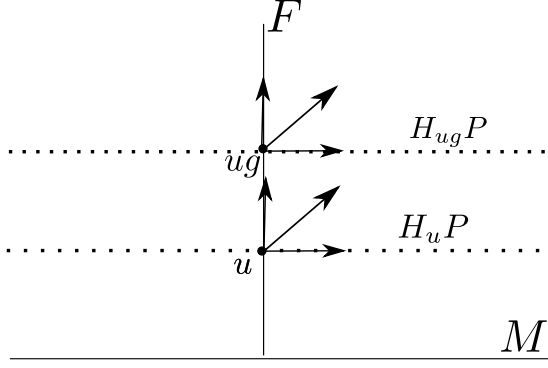


FIG. 2. A vector in the total space P shown decomposed into its vertical and horizontal components. The vector is additionally shown right translated "up" through the bundle.

extend over the manifold. Just as any section is a right translation of the canonical local trivialization, horizontal subspaces are related by a right translation. Splitting the tangent space in this way specifies a connection on P .

In order to extract the more familiar notion of connection in gauge field theory we define the fundamental vector field, $A^\#$, at a point $u \in P$ generated by $A \in \mathfrak{g}$ as,

$$A^\# f(u) = \frac{d}{dt} f(u e^{At})|_{t=0}, \quad (2)$$

for any function $f : P \rightarrow \mathbb{R}$. The vector $A^\#$ is directed entirely along the fiber. This can be seen by pushing forward a vector $X \in V_u P$ by the projection π as $\pi_* X$. With this the connection 1-form $\omega \in \mathfrak{g} \otimes T_u^* P$ is defined so that,

1. $\omega(A^\#) = A$ for $A \in \mathfrak{g}$,
2. $R_g^* \omega = g^{-1} \omega g$.

The horizontal subspace can then be defined as the kernel of ω ,

$$H_u P = \{X \in T_u P | \omega(X) = 0\}. \quad (3)$$

Now that we have a connection on the total space we can use a section σ to pullback the connection form ω to the base. The connection form on the base can be written,

$$\mathcal{A}_i = \sigma_i^* \omega. \quad (4)$$

The connection 1-form ω is a global quantity defined in the total bundle. We often work in terms of locally measurable quantities, e.g. the strength of the electromagnetic field in a local region of space. It is then more useful to start with a local connection 1-form \mathcal{A} on $U_i \subset M$ on the base and define the connection 1-form ω as,

$$\omega = g_i^{-1} \pi^* \mathcal{A}_i g_i + g_i^{-1} dg_i. \quad (5)$$

In the language of bundles a gauge transformation of a principal fiber bundle is a base preserving fiber automorphism [7, 8]. That is $f : G \rightarrow G$ such that $R_g \circ f = f \circ R_g$

and $f \circ \pi = \pi$. The first condition says that f must commute with all right actions of G and so is a left action of G , L_g [7]. The second condition means that the transformation is directed vertically along the fiber and ensures that the gauge transformation does not induce a diffeomorphism of the base. If we write $f(u) = u\eta(u)$ for $u \in P$ and $g \in G$ and $\eta : G \rightarrow G$ defined as $\eta(u) = u^{-1}gu$ we can determine the effect of a gauge transformation on a vector $X \in T_u P$. Let $\gamma : \mathbb{R} \rightarrow P$ such that $\gamma(0) = u$ and $\dot{\gamma}(0) = X$ (where the dot represents a parameter derivative) we find,

$$f_* X = \frac{d}{dt} f(\gamma(t)) \Big|_{t=0} \quad (6a)$$

$$= \frac{d}{dt} \gamma(t) \eta(\gamma(t)) \Big|_{t=0} \quad (6b)$$

$$= \left(\frac{d\gamma}{dt} \eta + \gamma \frac{d\eta}{dt} \right) \Big|_{t=0} \quad (6c)$$

$$= R_{g*} X + f(\gamma) \eta^{-1}(\gamma) \frac{d\eta}{dt} \Big|_{t=0} \quad (6d)$$

$$f_* X = R_{g*} X + (\eta^{-1} d\eta)^\#(X). \quad (6e)$$

Equation 6a is the definition of the pullback of a vector. In the next line we have used the definition of f . In Equation 6d we have rewritten the first term as the pushforward of the vector X . In the second term of Equation 6d we replaced $\gamma = f(\gamma) \eta^{-1}(\gamma)$. In the final step the second term can be seen to represent the fundamental vector field at $f(u)$ via Equation 2,

$$\begin{aligned} (\eta^{-1} d\eta)^\# f(u) &= \frac{d}{dt} f(u e^{\eta^{-1} d\eta}) \Big|_{t=0} \\ &= f(u) \eta^{-1} d\eta = f(u) \eta^{-1} \frac{d}{dt} \eta(\gamma) \Big|_{t=0}. \end{aligned} \quad (7)$$

Once we have the action of a gauge transformation on a vector in the total space, we can apply the connection 1-form ω and pullback the result to the base space. Doing this we obtain the transformation properties of the local connection 1-form \mathcal{A} . Using Equation 6e let $\sigma_i : M \rightarrow P$ be a local section over the subset $U_i \subset M$. Applying ω to Equation 6e we find,

$$f^* \omega(X) = R_g^* \omega(X) + \omega((\eta^{-1} d\eta)^\#(X)). \quad (8)$$

Applying σ_i^* to pullback the gauge transformation of the connection 1-form ω to the gauge transformation of the local connection 1-form \mathcal{A} we have

$$f^* \mathcal{A}_i = g^{-1} \mathcal{A}_i(X) g + \eta^{-1} d\eta(X). \quad (9)$$

For the example of $M = \mathbb{R}^{1,3}$ and $G = U(1)$ we coordinatize the space and expand the local connection in the dual basis as $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. The exterior derivative then takes the familiar form when acting on functions (say $\theta \in \mathcal{F}(M)$), $d\theta = \partial_\mu \theta dx^\mu$,

$$\begin{aligned} f^* \mathcal{A}_i &= \mathcal{A}'_i = \mathcal{A}_i + e^{-i\theta} d(e^{i\theta}) \\ \mathcal{A}'_{i\mu} &= \mathcal{A}_{i\mu} + i \partial_\mu \theta. \end{aligned} \quad (10)$$

Equation 10 is the familiar gauge transformation of electromagnetism, and so we see that vertical bundle automorphisms are indeed the correct notion of gauge transformations for internal symmetry groups.

The field strength tensor in the fiber bundle language is the pullback of the curvature 2-form Ω . To define this quantity we first define the covariant derivative of an r-form $\xi \in \Omega^r(P)$ as,

$$D\xi(X_1 \cdots X_r) = d\xi(X_1^H \cdots X_r^H), \quad (11)$$

where $X_1 \cdots X_r \in T_u P$. The curvature 2-form can then be defined as the covariant derivative of the connection 1-form,

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]. \quad (12)$$

The commutator of a \mathfrak{g} valued p-form ξ and q-form η is given as,

$$[\xi, \eta] = \xi \wedge \eta - (-1)^{pq} \eta \wedge \xi = [w_\alpha, w_\beta] \otimes \xi^\alpha \wedge \eta^\beta, \quad (13)$$

where we have decomposed the forms in the Lie algebra as $\xi = \xi^\alpha \otimes w_\alpha$. Setting $\eta = \xi$ we have,

$$[\xi, \xi] = 2\xi \wedge \xi = [w_\alpha, w_\beta] \xi^\alpha \wedge \xi^\beta. \quad (14)$$

We can use Equation 14 in Equation 12 to write the curvature in a more useful way. First we calculate the second term in Equation 12,

$$\begin{aligned} [\omega, \omega](X, Y) &= [T_\alpha, T_\beta] \omega^\alpha \wedge \omega^\beta(X, Y) \\ &= 2[T_\alpha, T_\beta] \omega^\alpha(X) \wedge \omega^\beta(Y) \\ &= 2[\omega(X), \omega(Y)]. \end{aligned} \quad (15a)$$

In the process we have relabeled indices and used properties of the commutator. Using Equation 15a and Equation 14 in Equation 12 we can re-express the curvature as,

$$\begin{aligned} D\omega(X, Y) &= d\omega(X, Y) + [\omega(X), \omega(Y)] \\ &= (d\omega + \omega \wedge \omega)(X, Y). \end{aligned} \quad (16)$$

The curvature, like the connection, of the bundle is a global quantity in the total space. In a subset of the base space a local section can be chosen and used to pullback the curvature to the base space,

$$\mathcal{F}_i = \sigma_i^* \Omega. \quad (17)$$

Using the fact that $\sigma_i^*(\omega \wedge \omega) = \sigma_i^* \omega \wedge \sigma_i^* \omega$ we find the local form of the bundle curvature,

$$\mathcal{F}_i = d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i. \quad (18)$$

In the case of $U(1)$ the wedge product drops out leaving,

$$\mathcal{F} = d\mathcal{A} = (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu. \quad (19)$$

The components of Equation 19 are the familiar field strength of electromagnetism.

Matter fields are often accommodated by the introduction of a vector bundle associated with $P(M, G)$. For a principal bundle $P(M, G)$ we define a vector space V such that G has left action on V under a representation $\rho : G \rightarrow GL(n, K)$, with K the field underlying V . If we assign an action on the product space $P \times V$ as,

$$(u, \nu) \rightarrow (ug, \rho(g^{-1})\nu), \quad (20)$$

the associated vector bundle is the equivalence class of points where we identify all elements of the form $(u, \nu) \sim (ug, \rho(g^{-1})\nu)$. We denote the vector bundle associated to P by $E = P \times_G V$ or $E(M, F, G, \pi_E, P)$. The equivalent notion of the space of equivariant vector valued functions on P denoted by $C(P, V)$ is much simpler to work with. If we take $\tau \in C(P, V)$ the condition of equivariance is $\tau(ug) = \rho(g^{-1})\tau(u)$, exactly the condition by which we quotient $P \times V$ to construct $E = P \times_G V$ [15].

In terms of the associated bundles a matter field is a section $\xi : M \rightarrow E$ of the bundle $E = P \times_G V$ or is simply an element of the space $C(P, V)$ [15]. To obtain the covariant derivative we consider a curve that has been horizontally lifted into the bundle. A curve $\gamma' : \mathbb{R} \rightarrow P$ is horizontally lifted if $\pi \circ \gamma' = \gamma$ is a curve in the base space and the tangent vector to γ' always belongs to the horizontal subspace of P . Picking a curve which trivializes as $\phi^{-1}(\gamma') = (\gamma(t), e)$ we can represent a lifted curve by $\tilde{\gamma} = \gamma'(t)g(\gamma'(t))$ for some $g \in G$. For some $\psi \in C(P, V)$ its covariant derivative in the direction of X is given by,

$$\nabla_X \psi = \frac{d}{dt} \psi(\tilde{\gamma}(t)) \Big|_{t=0}. \quad (21)$$

Inserting the definition of $\tilde{\gamma}$ into Equation 21 we arrive at the covariant derivative as follows,

$$= \frac{d}{dt} \psi(\gamma'(t)g(\gamma')) \Big|_{t=0} \quad (22a)$$

$$= \frac{d}{dt} g^{-1}(\gamma') \psi(\gamma') \Big|_{t=0} \quad (22b)$$

$$= \frac{d}{dt} g^{-1} \psi(\gamma') + g^{-1} \frac{d}{dt} \psi \Big|_{t=0} \quad (22c)$$

$$= -g^{-1} \frac{dg}{dt} g^{-1} \psi(\gamma') + g^{-1} \frac{d}{dt} \psi \Big|_{t=0} \quad (22d)$$

$$= \left(-\mathcal{A}(X) \psi(\tilde{\gamma}) + \frac{d\psi(\tilde{\gamma})}{dt} \right) \Big|_{t=0}. \quad (22e)$$

These are the standard elements utilized in formulating gauge theory from the functional approach. One need simply specify an invariant action using the covariant derivative acting on sections of associated vector bundles using the pullback of the connection one-form, and the local expression of its curvature to form a kinetic term for the gauge field.

The fiber bundle formalism provides an underpinning for gauge theories that is particularly useful on topologically nontrivial spaces. The preceding analysis can be applied to any gauge theory of internal symmetry groups.

However for external or spacetime symmetry groups, especially those expected to be relevant for gravitation, the standard fiber bundle formalism must be generalized.

III. COMPOSITE BUNDLES AND GAUGE THEORIES OF GRAVITATION

Poincaré gauge theory is based on the group $ISO(1,3) = SO(1,3) \ltimes \mathbb{R}^{1,3}$, i.e. the semi-direct product of the Lorentz group and the translations. The connection splits into a direct sum of two components [16], the spin connection ω and the coframe field θ . The spin connection arises from the Lorentz symmetry and the coframe field from the translational symmetry. As a result there are two conserved quantities, the energy-momentum and spin-angular momentum currents, which arise from variations of the action with respect to the gauge degrees of freedom. The energy-momentum current couples to the curvature of the Lorentz connection and the spin-angular momentum current couples to the curvature of the translational connection. Both connections transform as proper connections,

$$A' = g^{-1}Ag + g^{-1}dg, \quad (23)$$

under Poincaré transformations. At first glance, torsion aside, this appears to be a satisfactory gauge theory containing Einstein's general relativity. However there are two significant criticisms of this approach. First the dual coframe has one contravariant (upper) Lorentz index, thus the dual frame transforms as a contravariant vector under Lorentz rotations. This can be determined by recalling that for $\Lambda \in SO(1,3)$ the Minkowski metric is invariant $\Lambda^i_a \Lambda^j_b \eta_{ij} = \eta_{ab}$, hence

$$ds^2 = \hat{\theta}^i \hat{\theta}^j \eta_{ij} = \hat{\theta}^a \hat{\theta}^b \Lambda^i_a \Lambda^j_b \eta_{ij} = \hat{\theta}^a \hat{\theta}^b \eta_{ab}. \quad (24)$$

This is not the case in Poincaré gauge theory where the coframe is a connection and transforms as such. This is a problem with the standard approach of Poincaré gauge theory. Either the coframe field is not a connection or the standard approach is not sensitive enough to detect the proper transformation properties.

The second criticism concerns the actions of the symmetry groups. In general relativity the symmetry groups are external isometry groups, i.e. they act on spacetime. In the previous section we worked with gauge transformations that were defined as vertical bundle automorphisms, or in other words transformations that did not move between points in the base spacetime. This was noticed by Lord in [7], where he proposed that spacetime gauge theory be based on the bundle $P(G/H, H)$ with H taken to be the Lorentz group, G taken to be the Poincaré group and the verticality condition $f \circ \pi = \pi$ be relaxed. Without the verticality condition gauge transformations could induce transformations on the base spacetime. The translational component of Poincaré gauge theory acts to move between points in the base space, hence dropping the verticality condition seems to be a step in the right

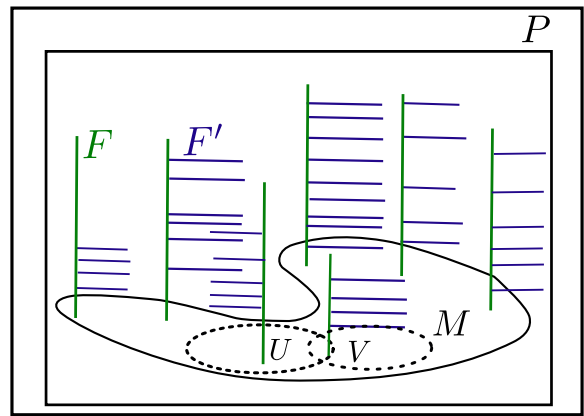


FIG. 3. A depiction of a composite bundle is shown. Like ordinary fiber bundles there is a base space M and fiber spaces F for each point of M . A point $u \in H$ in the total space created by the base M and fiber space F is decomposed locally as $u = (x, f) \in M \times F$. However for a composite bundle we have additional fiber spaces attached at each point $u \in H$. The total space of the composite bundle is then locally $E \cong H \times F'$ and can be decomposed further as $(x, f, f') \in M \times F \times F'$.

direction. However without a translational fiber space the theory lacked a translational connection.

The composite bundle theory of gravitation (detailed below) introduced by Tresguerres alleviates these problems by considering a principal G bundle over M , $P(M, G)$, to be split into a chain of bundles $P \xrightarrow{\pi_{PE}} \Sigma \xrightarrow{\pi_{EM}} M$ (Fig. 3) [8]. A consequence of using the composite scheme in the gravitational context is that the tetrad defined from the coframe can be identified as a nonlinear translational connection. This is closely related to the work of Coleman, Wess and Zumino [17] on nonlinear sigma models. In fact the composite bundle theory of gravitation is a geometric realization of nonlinearly realized symmetry groups. Julve et al. demonstrated the connection between gravitational gauge theory and nonlinearly realized gauge symmetries [18] and Tresguerres and Tiemblo explored the connection in the context of composite bundle formulations of gravitation [19].

In what follows we present a short exposition of the composite bundle approach introduced by Tresguerres. We will then discuss a crucial aspect of the composite formalism that we believe has not been addressed in the literature. This will lead us to consider the topological restrictions arising in composite bundle formulations of gravitation.

Much of the analysis of composite bundles is very similar to the internal case developed in section II. However now we will have two bundles over the base space M which will be related. The idea is to work with the principle bundle $P(M, G(G/H, H))$ where $G(G/H, H)$ itself denotes a principal fiber bundle with G/H as the base space and H as the fiber space. We can take sections as before only now there are three possible choices: one for each sector ($\sigma_{ME} : M \rightarrow E$ and $\sigma_{EP} : E \rightarrow P$), and one

for the total space ($\sigma : M \rightarrow P$). Consistency requires that $\sigma = \sigma_{EP} \circ \sigma_{ME}$. Sections can be written as before in terms of the local canonical trivialization,

$$\sigma_{ME}(x) = \phi_{ME}(p, e_{G/H})a(\xi) \quad (25a)$$

$$\sigma_{EP} = \phi_{EP}(u_E, e_H)h, \quad (25b)$$

where we have the identity elements $e_{G/H} \in G/H$ and $e_H \in H$ and the elements $u_E \in E$, $a \in G/H$ and $h \in H$. Formally we write u_E for the input to the section $\sigma_{EP} : E \rightarrow P$. However we have in mind that we are using a local section. Locally the bundle $E \rightarrow M$ can be seen as $M \times G/H$ and so we can specify a point $u_E \in E$ in Equation 25b as (p, ξ) .

Gauge transformations in the total bundle P respect the conditions we imposed in the previous section, i.e. they commute with the right action and are vertical with respect to the fiber. However in the sector $P \rightarrow E$ the verticality condition is relaxed. The gauge transformations in $P \rightarrow E$ are allowed to translate within the base space E . To see this we write the gauge transformation as $f = L_g$ for some $g \in G$ and $h \in H$,

$$f(\sigma_{EP}(p, \xi)) = \sigma_{EP}(p, \xi')h. \quad (26)$$

We consider the gauge transformations of a vector X tangent to a curve passing through the point $u = \sigma_{EP}(p, \xi)$ to find an equation analogous to Equation 6e,

$$f_*X = Q_*(R_{h*}X + (h^{-1}dh)^\#(X)). \quad (27)$$

There are two differences between Equation 6e and Equation 27. First although we used an element $g \in G$ as our left action, only the component $h \in H$ survives in the expression for the transformation. Second there is an additional pullback by $Q = R_h^{-1} \circ L_g$ present in the transformation. These differences will conspire to ensure the coframe transforms as a contravariant Lorentz vector. To see this we first need to introduce the connections. There appear to be two bundles on which we can put a connection, i.e. $P \xrightarrow{\pi_{PE}} E$ and $E \xrightarrow{\pi_{EM}} M$. But since the overall bundle is $P \rightarrow M$, there is really only one connection 1-form ω . The two connection 1-forms, $P \xrightarrow{\pi_{PE}} E$ and $E \xrightarrow{\pi_{EM}} M$, are the "shadow" of this connection pulled back to their respective base spaces,

$$\mathcal{A}_M = \sigma_{MP}^* \omega, \quad \mathcal{A}_E = \sigma_{EP}^* \omega. \quad (28)$$

The connection on E can be further pulled back to the base space M so that the two "shadow" connection 1-forms satisfy [8],

$$\mathcal{A}_M = \sigma_{ME}^* \mathcal{A}_E = \sigma_{ME}^* \sigma_{EP}^* \omega. \quad (29)$$

Over the bundle sector $P \rightarrow E$ we can split the total connection into a sum of two components $\omega = \omega_R + \omega_T$, the subscript R denotes Lorentz rotations and the subscript T denotes translations. Analogous to Equation 5 we have,

$$\omega_R = h^{-1}(d + \pi_{PE}^* \mathcal{A}_R)h, \quad (30)$$

where \mathcal{A}_R is the local form of the connection of the Lorentz connection on the base space E . Additionally we have,

$$\omega_T = h^{-1} \pi_{PE}^* \mathcal{A}_T h. \quad (31)$$

Where \mathcal{A}_T denotes the local form of the translational connection on E . Applying the total connection ($\omega = \omega_R + \omega_T$) to Equation 27, pulling the result to the base space E and equating terms based on their expansions in the Lie algebras of the translations and Lorentz rotations we arrive at the gauge transformation of the gauge fields,

$$h^{-1}dh + h^{-1} \mathcal{A}_R h = \mathcal{A}'_R, \quad h^{-1} \mathcal{A}_T h = \mathcal{A}'_T. \quad (32)$$

The infinitesimal variation can be computed from $\delta A = A - A'$ where A' is the gauge transformed form of A . We expand the group transformations $h = e^{i\Lambda_{\alpha\beta}\epsilon^{\alpha\beta}} \approx I + i\Lambda_{\alpha\beta}\epsilon^{\alpha\beta}$ to arrive at,

$$\delta \mathcal{A}_T^i = -\mathcal{A}_T^j \epsilon^i_j. \quad (33)$$

This is exactly the infinitesimal variation of a Lorentz transformation. Breaking the bundle $P \rightarrow M$ to $P \rightarrow E \rightarrow M$ gives the right transformation properties of \mathcal{A}_T while leaving it to still be identified as a gauge potential [8]. This gauge potential (\mathcal{A}_T) when pulled back to the base by a canonical local trivialization $\sigma_{ME}(p) = \phi(p, e)$ is identified as $\sigma_{ME}^* \mathcal{A}_T = \hat{\theta}$, i.e. the coframe. We can decompose the coframe as $\hat{\theta}^i = e^i_\mu dx^\mu$ and so can write the metric as,

$$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}. \quad (34)$$

A further indication that composite gauge theories describe gravitation comes from expanding the tetrad in terms of the spacetime connection [8],

$$e_\mu^i = \partial_\mu \xi^i + \mathcal{A}_{R\mu j}^i \xi^j + \mathcal{A}_{T\mu}^i. \quad (35)$$

We can see that the tetrad has internal structure in terms of the dynamic gauge fields as expected since the Christoffel connection was defined in terms of the metric.

To determine whether composite bundles provide a consistent formulation of gravitation, we should stipulate what sort of objects we expect the construction to reproduce. This point has been slighted in the literature. The requirements arise from considering what bundles are relevant to general relativity. To formulate general relativity via a gauge principal we must show how the tangent bundle and the frame bundle arise as a consequence of local symmetry. The tangent bundle is a vector bundle over a manifold M denoted $TM \equiv (M, \mathbb{R}^n, Gl(n, \mathbb{R}))$. Its structure group is the general linear group and its fiber space is just n dimensional Euclidean space. Associated with the tangent bundle is the frame bundle, a principal fiber bundle denoted $\mathcal{FM} \equiv (P(M, Gl(n, \mathbb{R})))$. So the tangent bundle is the associated vector bundle to the frame bundle. A Lorentzian metric on a space M is an inner product operation on

the tangent bundle $\eta : TM \otimes TM \rightarrow \mathbb{R}$. Introducing a Lorentzian metric on the tangent bundle reduces the structure group of the tangent bundle and the frame bundle from the general linear group to the orthogonal group, $Gl(n, \mathbb{R}) \rightarrow O(1, n-1)$ (supposing that there is one timelike dimension and $n-1$ spatial dimensions). So we are left with $TM = E(M, \mathbb{R}^n, O(1, n-1))$ and $\mathcal{F}M = P(M, O(1, n-1))$. We can reduce the symmetry group and further extend it as we continue to remove topological obstructions. This will be the topic of the following section. However we are still left with the question “how do the frame bundle and its associated vector bundle arise in the context of composite gauge theories of gravitation?”.

To answer this question we need to go back to how and when can we split a principal bundle into a “tower” of bundles. The basic idea, as Tresguerres applies it, comes from proposition 5.5 and 5.6 of Kobayashi and Nomizu. Proposition 5.6 states “The structure group G of $P(M, G)$ is reducible to a closed subgroup H if and only if the associated bundle $E(M, G/H, G, P)$ admits a cross section $\sigma : M \rightarrow E = P/H$ ” [16]. We are able to identify $E = P/H$ due to proposition 5.5 which states, “The bundle $E = P \times_G G/H$ associated with P with standard fiber G/H can be identified with P/H as follows. An element of E represented by $(u, a\xi_0) \in P \times G/H$ is mapped into the element of P/H represented by $ua \in P$ where $a \in G$ and ξ_0 is the origin of G/H , i.e., the coset H .” [16]. This proposition assures us that the associated bundle E can be seen to be exactly the total bundle P quotiented by the closed subgroup H . When reading these propositions it is easy to use an everyday reading of the word reducible. However reducible in these propositions has a very specific meaning. Kobayashi and Nomizu say that provided there exists an embedding $f : P'(M, G') \rightarrow P(M, G)$ then the image $f(P')$ is a sub-bundle and we say G is reducible to G' [16]. An understanding of what reduced means is crucial for the identification of the topological structures of classical spacetime. The global section of E leads to a subbundle of $P(M, G)$ given by,

$$Q(M, H) \equiv \{u \in P(M, G) | \pi_{PE}(u) = \sigma(x)\}. \quad (36)$$

Since the section σ is global and so too is π_{PE} we can define a new global section given by $q \equiv \pi_{PE}^{-1} \circ \sigma : M \rightarrow Q$. We will use this section to understand the topology of spacetime induced by the composite bundle structure. There is a further bundle we will need. For every principal bundle there is an associated vector bundle. So we can also construct $Q' = Q \times_H \mathbb{R}^n$. As a result of requiring a global section to split $P(M, G)$ we have a collection of bundles to work with which are displayed in Table I. In this collection of bundles we find the appearance of the frame and tangent bundle in composite gauge theory. For $G = ISO(1, 3)$ the Poincaré group, $H = SO(1, 3)$ the Lorentz group and $G/H = \mathbb{R}^4$ the bundle Q is diffeomorphic to the frame bundle $\mathcal{F}M \cong Q(M, SO(1, 3))$. The associated vector bundle to Q is diffeomorphic to the tangent bundle $Q' = Q \times_{SO(1, 3)} \mathbb{R}^4 \cong TM$. The global

TABLE I. The collection of bundles formed during a composite bundle construction and their projections.

Bundle	Projection
$P(E, H)$	π_{PE}
$E(M, G/H, G, P)$	π_{EM}
$Q(M, H)$	μ
$Q(M, H) \times_H \mathbb{R}^n$	μ_A

sections of the bundle E , which assure us a subbundle $Q \subset P$, also act to connect the translation and rotational gauge degrees of freedom of $P(E, H)$ and $E \cong P/H$ to the spacetime bundles $\mathcal{F}M$ and TM .

Another natural question to ask is, “over which manifolds can we find global sections of E ?”. Theorem 5.7 of Kobayashi and Nomizu gives that if the base space M is paracompact and the fiber space, here $F = G/H$ is diffeomorphic to an Euclidean space any section defined over a closed subset of M can be extended to the entire space M [16]. In the following section we will discuss why this theorem will hold for the composite bundle we have constructed for gravitation. Additionally using all of the bundles in Table I we will see that a consequence of demanding that we can break the bundle $P \rightarrow M$ to $P \rightarrow E \rightarrow M$ is that the induced tangent and frame bundles of M must have a set of trivial characteristic classes.

IV. TOPOLOGICAL OBSTRUCTIONS

The construction of a consistent composite bundle requires that the associated bundle E admit a global section σ . A natural question to ask is on which base spaces can the bundle E have a global section? And more importantly what does this imply for the induced tangent and frame bundles? Not every manifold will admit such a topological structure. We therefore need a way to classify the topological obstruction to a global section.

In the case of composite bundles some of the obstructions reside in the Čech cohomology. This cohomology is built entirely from the transition functions of a bundle. A bundle is trivial iff we can choose every transition function to be trivial. So it is natural to suspect a cohomology built from the transition functions to be an appropriate tool to understand the consequences of global sections of $E(M, G/H, G, P)$ for the topology of the induced spacetime bundles.

To build the Čech cohomology we follow the same general steps as any cohomology theory and follow closely [13]. The coefficients of the Čech cohomology are \mathbb{Z}_2 , which we take to be the multiplicative group of elements $\{-1, 1\}$. Analogous to the cocycle group in the de Rham cohomology we define the Čech r-cochain as a function $f(t_0, \dots, t_r) \in \mathbb{Z}_2$ defined on the set $U = \cap_{j=0}^r U_j$ where the t_r are the transition functions. Additionally we require f be invariant under arbitrary permutation. Let C^r denote the multiplicative group of Čech

r-cochains. Then we also require a nilpotent operator $\delta : C^r(M) \rightarrow C^{r+1}(M)$ defined as,

$$\delta(f(t_0, \dots, t_r)) = \prod_{k=0}^{r+1} f(t_0, \dots, \hat{t}_k, \dots, t_{r+1}), \quad (37)$$

where the "hat" above the k^{th} entry denotes removal of that quantity. For example consider a 2-cochain f . If we allow the boundary operator to act on f we find the following 3-cochain,

$$\delta(f(t_0, t_1)) = f(t_1, t_2)f(t_0, t_2)f(t_1, t_2). \quad (38)$$

We can see that δ is a nilpotent operation, i.e. applying δ twice we will be removing two elements,

$$\delta(\delta f) = \prod_{j,k=0}^{r+1} f(t_0, \dots, \hat{t}_k, \dots, \hat{t}_j, \dots, t_{r+1}). \quad (39)$$

Note that every combination will appear twice since f is invariant under arbitrary permutations,

$$\begin{aligned} f(t_0, \dots, \hat{t}_k, \dots, \hat{t}_j, \dots, t_{r+1}) \\ = f(t_0, \dots, \hat{t}_j, \dots, \hat{t}_k, \dots, t_{r+1}). \end{aligned} \quad (40)$$

Since the values f takes are in \mathbb{Z}_2 we have $f^2 = 1$ and so we see δ is nilpotent,

$$\delta^2(f(t_0, \dots, t_r)) = 1. \quad (41)$$

If we let $Z^r(M; \mathbb{Z}_2) = \{f \in C^r(M) | \delta f = 1\}$ be the cocycle group and $B^r(M; \mathbb{Z}_2) = \{f \in C^r(M) | f = \delta h, h \in C^{r-1}\}$ the coboundary group, we define the Čech cohomology as,

$$H^r(M; \mathbb{Z}_2) = \frac{Z^r(M; \mathbb{Z}_2)}{B^r(M; \mathbb{Z}_2)}. \quad (42)$$

An element $f \in Z^r(M, \mathbb{Z}_2)$ defines a class $[f] \in H^r(M, \mathbb{Z}_2)$ given by the set in Equation 43,

$$\{f' \in Z^r(M, \mathbb{Z}_2) | f' = \delta h f, h \in C^{r-1}(M, \mathbb{Z}_2)\}. \quad (43)$$

Our initial focus is on the Stiefel-Whitney classes w_r which are elements of the Čech cohomology classes. The non-triviality of these classes are obstructions to the creation of certain structures on bundles. It is well known that a non-vanishing first Stiefel-Whitney class is an obstruction to orientability [13, 20, 21]. Let M be a manifold and $TM \xrightarrow{\pi} M$ be its tangent bundle. The Stiefel-Whitney classes of M are the same as the those of its tangent bundle. Proof of this statement can be found in [20]. So the Stiefel-Whitney classes of a manifold are essentially the classes of its tangent bundle.

Let TM have a Riemannian structure provided by $g : \pi^{-1}(p) \otimes \pi^{-1}(p) \rightarrow \mathbb{R}$. Recall that $\pi^{-1}(p) \cong F$, i.e. the Riemannian structure on TM is then a mapping $g : TM \otimes TM \rightarrow \mathbb{R}$. We see a Riemannian structure on the tangent space is the equivalent of a metric. In general relativity we could choose a local frame as a

rotation of the natural basis in the tangent space. If we consider the set of all of these choices we can build a principal fiber bundle called the frame bundle. The frame bundle has the tangent bundle as an associated bundle and is written $\mathcal{FM}(M, Gl(n, \mathbb{R}))$. At first we have a choice of $e^\mu_i \in Gl(4, \mathbb{R})$. However we require that the frame \hat{e} be orthonormal with respect to the metric. This requirement reduces our set of choices from a $Gl(n, \mathbb{R})$ rotation to an $O(n)$ (or $O(1, 3)$ for Lorentzian spacetimes) rotation. The reduction in structure group means on overlapping open neighborhoods $U_i \cap U_j$ that a choice of frame over U_i is related to the choice in frame of U_j by $e_\alpha = t_{ij} e_\beta$ with $t_{ij} \in O(n)$ as the transition function. Let f be the determinant function, we have $f(t_{ij}) = \pm 1$, i.e. f is indeed valued in \mathbb{Z}_2 . On a triple intersection $U_i \cap U_j \cap U_k$ the transition functions (t_{ij}, t_{jk}, t_{ki}) are required to satisfy the cocycle condition $t_{ij} t_{jk} t_{ki} = I$ [13, 14]. Using the cocycle condition we can act the boundary operator on our cochain function to find $\delta f(i, j) = 1$, so $f \in Z^1(M; \mathbb{Z}_2)$ and defines an equivalence class $[f] \in H^1(M; \mathbb{Z}_2)$. This first class $w_1 = [f]$ is the first Stiefel-Whitney class. It follows that M is orientable iff w_1 is trivial, see [13] for proof.

The second Stiefel-Whitney class w_2 is an obstruction to a bundle admitting a spin structure. Defining spinors on a space requires a lifting to the covering group of $SO(n)$ (or the covering group of $SO(1, 3)$ for 4-dimensional Lorentzian spacetimes) and the second Stiefel-Whitney class describes the obstruction to defining this lifting. Since we can reconstruct a bundle from its transition functions, consider a manifold M whose tangent bundle is orientable i.e. the first Stiefel-Whitney class is trivial, and let $\{t_{ij}\}$ be the set of transition functions of the associated principal frame bundle \mathcal{FM} . If we let $\psi : SPIN(n) \rightarrow SO(n)$ be the typical double covering of the group $SO(n)$ we can define a complementary set of transition functions $\{\tilde{t}_{ij}\}$ such that $\psi(\tilde{t}_{ij}) = t_{ij}$ and,

$$\tilde{t}_{ij} \tilde{t}_{jk} \tilde{t}_{ki} = I, \quad (44)$$

on a triple intersection $U_i \cap U_j \cap U_k$. The set of $\{\tilde{t}_{ij}\}$ defines a spin bundle over M denoted $SPIN(M)$ and M is said to admit a spin structure. Consequently we have that a manifold M admits a spin structure iff the second Stiefel-Whitney class vanishes [13].

For our purposes there is a more useful relation from Flagga and Antonsen. In [22, 23] the authors make a case for the topology of spacetime. Their papers center around the interpretations of the third and fourth Stiefel-Whitney classes. They argue that the interpretations of the remaining classes are related to obstructions to chiral spinors and causality respectively (Table. II displays the interpretations of each Stiefel-Whitney class). Additionally [22] contains a useful list of equivalent statements for a bundle to admit a spin structure. From their list an equivalent statement to the vanishing of the second Stiefel-Whitney class is, "an orientable manifold M admits a spin structure iff M is parallelizable" [22]. Parallelizability is the ability to define a global section $\sigma : M \rightarrow \mathcal{FM}$ of the frame bundle, i.e. the frame bundle

TABLE II. A list of the Stiefel-Whitney classes and their interpretations as obstructions to topological entities over a manifold M .

Stiefel-Whitney class	What is it obstructing?
w_1	Orientability
w_2	Spin
w_3	Chiral Spinors
w_4	Causality

must be trivial. For us to use this fact we must recall the two equivalent ways of deciding the triviality of a vector bundle. We can either find an everywhere non-zero global section of the vector bundle or we can find a global section of the associated principal bundle. This means that showing that we can find a section of $E(M, F, G, P)$ means we can find a section of $P(M, G)$. Looking to Milnor and Stasheff, “if the oriented vector bundle ξ possesses a nowhere zero cross-section, then the Euler class $e(\xi)$ must be zero” [20].

Now that we have collected most of the characteristic classes needed we can discuss how this applies to composite bundles. Consider first that the base manifold we work with is compact. Then we start with the total bundle $P(M, G)$ where we interpret the Poincaré group ($G = ISO(1, 3)$) as the bundle $G(G/H, H)$ for $G/H \cong \mathbb{R}^4$ the translations and $H = SO(1, 3)$ the Lorentz rotations. Then we can construct the bundle $E(M, G/H, G, P) = P \times_G G/H \cong P/H$ associated to $P(M, G)$. From III we know that there exists a bundle $P(E, H)$ with E as the base space and H as the fiber space. Then provided that there exist a global section $\sigma : M \rightarrow P$ the structure group can be reduced from G to $H \subset G$ [16]. Additionally from section III we know that that this means there exists $Q(M, H) \subset P(M, G)$ and an associated bundle $Q(M, H) \times_H \mathbb{R}^4$. The bundle $Q(M, H)$ can be identified with the principal frame bundle over M , and its associated vector bundle $Q(M, H) \times_H \mathbb{R}^4$ can be identified as the tangent bundle to the base space M . By creating a principal Poincaré bundle we have indeed succeeded in creating the needed bundles for general relativity, i.e. a tangent bundle and a frame bundle. If we did not require a global section of $E \cong P/H$ then we would have no way of connecting the gauge bundle with the frame bundle meaning that the gauge bundle would not influence spacetime. This point seems to be missed in the literature. Global sections of E ensure communication with the tangent space of the base spacetime. The theorem used in section III guarantees global sections of this bundle in the case of paracompact manifolds, in a compact spacetime it is simple to see the extension of the theorem. The question now is what this implies of the topology of the base manifold. From section III we saw that the bundle Q was defined as $Q(M, H) \equiv \{u \in P | \pi_{PE}(u) = \sigma(x)\}$, leading us to define the global section in Q as $q = \pi_{PE}^{-1} \circ \sigma : M \rightarrow Q(M, H) \cong \mathcal{FM}$. We now have that the frame bundle of M is trivial and

as a consequence we can extend this global section to $q' : M \rightarrow Q(M, H) \times_H \mathbb{R}^4 \cong TM$ leading to a trivial tangent bundle. Putting a Riemannian structure on the tangent bundle we can then verify that the both the first and second Stiefel-Whitney classes are trivial as done above. Then as consequence of the second Stiefel-Whitney class being trivial the third Stiefel-Whitney class is also trivial [22]. With a tangent bundle that is orientable and possesses a global section we know that the Euler class of the tangent bundle must be zero. But we need to be a bit more precise, i.e. $e(M) = e(TM) \in H^4(M; \mathbb{Z})$. With integer coefficients the Euler class is carried by the natural homomorphism to the top Stiefel-Whitney class and so it too is trivial [20]. If instead we work with real coefficients in the de Rham cohomology the Euler class of an even dimensional manifold squares to the first Pontryagin class p_1 , requiring it to trivialize.

The spin structure on a manifold is consistently defined iff the second Stiefel-Whitney class is trivial. There is an analogous condition for a manifold to have a string structure, i.e. the first fractional Pontryagin class must trivialize. This means that not only must p_1 be trivial but the cohomology group must be torsion free [24]. For the case of compact base spaces we now are left with only the torsion decomposition piece.

A cohomology group H^r can be decomposed into two pieces, a free piece and a torsion piece. The vanishing of the first fractional Pontryagin class is the condition that $H^4(M; \mathbb{Z})$ be decomposed as only a free piece and the characteristic class $p_1 \in H^4(M; \mathbb{R})$ be trivial. We already saw that the first Pontryagin class is trivial. What is left to show is that the decomposition of the cohomology group has only a free piece. The idea here is that a finitely generated abelian group is the direct sum of a free abelian group of finite rank and a finite abelian group [25]. A free group has a basis in which we can represent an element in terms of the basis elements and for a group to be finitely generated there must be a finite number of basis elements. The finite abelian subgroup is the torsion subgroup. Loosely the torsion subgroup is made up of all elements with finite order. Depending on additional constraints we put on our base space we have two cases to consider. Provided that M is a connected and compact manifold, the homology groups are finitely generated. And provided there is an orientation (which there is), we can use a theorem from Hatcher [25] that “If M is a compact connected manifold of dimension n then $H_{n-1}(M, \mathbb{Z})$ is trivial if M is orientable.” In this case using the universal coefficient theorem and Poincaré duality we have $H^4(M, \mathbb{Z}) = H_4(M, \mathbb{Z}) \oplus H_3(M, \mathbb{Z})$. However $H_3(M, \mathbb{Z})$ is trivial so we have exact Poincaré duality, i.e. $H^4(M, \mathbb{Z}) = H_4(M, \mathbb{Z})$. Along with the previous result of the triviality of the integral cohomology class we conclude that the manifold M admits a lifting to $STRING(n)$.

The cohomology becomes more complicated when the manifold M is non-compact. As discussed above in [22, 23], the authors make a case for the topology of spacetime and work with non-compact manifolds. To work with a sensible cohomology theory on non-compact

spacetimes we need cohomology with compact support. If we were to work with a manifold M and define on some compact subset $U \subset M$ a function $\tilde{f} : M \rightarrow \mathbb{R}$ such that $\tilde{f}|_U = f : U \rightarrow \mathbb{R}$ and $\tilde{f}|_{M-U} = 0$, then we say that f has compact support. This idea will be the basis for compactly supported cohomology. We will have cochains that vanish outside of a compact set.

For a non-compact manifold we instead work with the open sets which cover M . To begin we first define a relative chain group. Suppose we have some topological space X and A a subspace of X . Then we have a subset of the chain group $C_n(A) \subset C_n(X)$. The boundary operator $\partial : C_n \rightarrow C_{n-1}$ acting on $C_n(A)$ is the restriction of ∂ to A and so gives back C_{n-1} . The relative chain group is then given by the quotient $C_n(X, A) = C_n(X)/C_n(A)$. The relative homology group is defined as before as a quotient of the kernel and image of the boundary map ∂ . The dual system is the relative cohomology $H^r(X, A; G)$ (for a finite abelian group G) and is defined as a quotient of the cochain and coboundary group as before only using relative version of the groups instead [25]. An element in $H^r(X, A; G)$ is a class which vanishes on a set $A \subset X$. We can immediately see the usefulness of this cohomology when the set A is replaced by its complement $X - A$. The cohomology $H^r(X, X - A; G)$ has elements which vanish outside of the set A , exactly the condition we need to define cohomology with compact support.

The final piece we need to build the cohomology with compact support for the composite gauge theory of gravity is a set of compact subsets of the base manifold M . Flagga and Antonsen accomplish this with the aide of the paracompactness of M and the fact that every metric space is normal. With these conditions, each locally finite cover \mathcal{U} has a refinement $\mathcal{U}' = \{U' | \text{for } m \in \mathcal{U}', \cup \{U' \in \mathcal{U}' | m \in U'\} \subset U\}$ [23]. A space which meets this criterion is called strongly paracompact [23]. Furthermore if each open cover has a countable subcover then the space is called finally compact and the countable subcover is called a shrinking [23]. With this information we can then find a sequence of compact sets,

$$k_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \cdots \subset \overset{\circ}{K}_j \subset K_j, \quad (45)$$

such that $\cup_{j \in \mathbb{N}} \overset{\circ}{K}_j = M$ where $\overset{\circ}{K}$ denotes the interior of the set K [23]. If we now do as above and create a cochain group $C^n(M, M - K_j, \mathbb{Z})$, then the compactly supported cochain group is $C_c^n = \cup_{j \in \mathbb{Z}} C^n(M, M - K_j, \mathbb{Z})$ [25]. Additionally we notice that the coboundary operator moves us within the relative cochain group since if $f \in C_c^n(M; \mathbb{Z})$ vanishes on $M - K_j$ then so does δf and so we can write the cohomology with compact support as,

$$H_c^r(M; \mathbb{Z}) = \frac{Z_c^r(M; \mathbb{Z})}{B_c^r(M; \mathbb{Z})}, \quad (46)$$

only now we work with the compactly supported cocycle and coboundary groups.

Since every spacetime we will consider in a physically realistic theory will have a Lorentzian metric and we do require the existence of a direction field to distinguish future and past directed events every reasonable

spacetime we encounter will be paracompact. This allows us to build a cohomology theory for the composite bundles even though we do not require the base space to be compact. Even with the compactly supported cohomology we still require the same conditions to hold in order to build a composite theory of gravitation, vanishing Stiefel-Whitney classes, vanishing Euler class and vanishing first Pontryagin class. We are left showing that $H^4(M; \mathbb{Z})$ can be decomposed into only a free part. To do so we would like to use the same lemma from [25], although there will be some differences. Hatcher's lemma requires compact connected spacetimes so that the homology groups are finitely generated. Thus to use the same reasoning we must have that the compactly supported homology groups we have created are finitely generated. If we can show this we are free to use his lemma. Fortunately we have that provided that a space M has a good cover then $\dim H_c^r(M; \mathbb{R}) < \infty$ [26]. With this result we can now use the Poincaré duality for compactly supported cohomology to say that the spaces H_c^r are isomorphic to H_{n-r} which is to say the dimension of the homology groups are finite for these spaces [21, 25]. Using lemma 3.27 from Hatcher which states that " $H_i(M, M - A; \mathbb{R}) = 0$ for $i > n$ on a compact subset A of an n dimensional manifold M " we see that there are finitely many homology groups [25]. With a finite number and finite dimension, our homology groups are finitely generated. Then using the universal coefficient theorem $H_r(M; \mathbb{R}) \cong H_r(M; \mathbb{Z}) \otimes \mathbb{R}$ and so the homology groups with integer coefficients are finitely generated. We can finally use the flavor of corollary 3.28 of [25] to say that $H_{r-1}(M; \mathbb{Z})$ has trivial torsion subgroup. The result of all of this formalism is that in the realistic case of a paracompact base space we also have the trivialization of first Pontryagin class and the fourth cohomology class has no torsion. In agreement with the conditions put forth by [22, 23] on a reasonable spacetime topology, in order to construct a composite bundle formulation of gravity we are forced to have the manifold admit a string structure.

That a consistent bundle underpinning of gauge theories of gravitation requires the underlying spacetime to admit a string structure comes as a surprise. Among the underlying conditions determining the consistency of perturbative theories are anomaly cancellations. However anomaly cancellations and topological obstructions have been associated before. For example the first and second Stiefel-Whitney classes must be trivial in order for a space to admit a spin structure. In the case of a supersymmetric point particle it can be shown that there is a global anomaly if the manifold over which the theory is defined does not admit a lifting to $\text{spin}(n)$. The relevance of lifting to the string group in physics was pointed out by Killingback [27]. He describes the role of the 2-form B and its 3-form field strength H in the cancellation of spacetime anomalies. The 3-form obeys,

$$dH = \text{Tr}(F^2) - \text{Tr}(R^2) \quad (47)$$

for F a Yang-Mills field strength and R the curvature 2-

form. Integrating this over a closed subset of base space M results in a consistency condition that the two Pontryagin classes of the tangent bundle of spacetime and of either $SO(32)$ or $E_8 \times E_8$, be equivalent.

If we can lift to spin, then the transition functions $(\{\tilde{t}_{ij}\})$ define a principal spin bundle $SPIN(M, Sl(2, \mathbb{C}^2) \oplus \overline{Sl(2, \mathbb{C}^2)})$ over M . In [28] Waldorf considers connections A on the bundle $SPIN(M)$ and defines a string class and further a geometric string structure as a pairing of a string structure and a connection on $SPIN(M)$. Studying the consequences Waldorf finds that under certain conditions there exists a 3-form H on M with the prescribed properties needed for anomaly cancellation as described above. Most importantly the exterior derivative of H “is one-half of the Pontryagin 4-form of A ” [28]. Although we did not specify that we needed a connection on the spin bundle over M , it is a natural construction to add to the bundle. When we do, we find that the composite theories of gravitation come naturally equipped with information needed for anomaly cancellation.

V. CONCLUSIONS

The verdict is still out on whether or how general relativity can be formulated as a gauge theory akin to our understanding of the Standard Model interactions. While much of the debate has been cast in terms of the functional formalism, the possibility of nontrivial spacetime topologies requires an underpinning in terms of some type of bundle structure. We feel that the composite bundles advocated by Tresguerres provide the best framework for accommodating both the local Lorentz and translational symmetries of the Poincaré group which itself seems to be an essential starting point in any gauge theory of gravitation. Pursuing the implications of this formulation for the structure of spacetime has led to some interesting results.

We have found that the composite gauge theory formulation of general relativity requires that the Stiefel-Whitney, Euler and first rational Pontryagin classes of spacetime be trivial. This set of topological restrictions is consistent with the conclusion that spacetime must be

a string manifold, i.e. admit a string structure. This comes as a surprise since this analysis has been purely classical and the usual motivation for extended degrees of freedom like strings (and hence the admittance of string structures for spacetime) in gravitational theories is to achieve quantum consistency. Perhaps this result might have been anticipated since every viable gauge theory of gravity seems to necessitate the presence of torsion in a manner reminiscent of the inevitable inclusion of the Kalb-Ramond field (whose field strength mimics torsion) in the massless multiplet that includes the graviton in string theory.

It should be stressed that we did not start out to investigate gauge theories of extended objects as in the program of so called higher gauge theory [11]. Instead our starting point was the minimal bundle formulation accommodating both the Lorentz and translational symmetries underlying Poincaré gauge theory. However given the resulting connection to extended degrees of freedom, it would be an interesting task to connect the composite bundle framework with developments in higher gauge theory.

Our analysis has largely focused on the simplest realization of general relativity and hence requires no more than four spacetime dimensions. In string theory, we not only encounter topological restrictions based on string structure, but also restrictions based on the admittance of a five-brane structure [24]. Clearly the five-brane structure is not relevant for four dimensional settings, but investigation of the consequences of composite bundle consistency in higher dimensions could very well lead to the requirement of five-brane structure as well. This topic is under current investigation.

Yet another consideration is the inclusion of internal gauge symmetries in the composite formulation. The consistent splitting of the total bundle into a composite structure could likely lead to related conditions between topological invariants of the base spacetime and of the internal gauge bundle. These results might very well tie in with the spacetime and gauge anomaly cancellation mechanisms underlying the consistency of Type I and heterotic strings theories just as in Equation 47.

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